MULTIPLICATIVE FUNCTIONS THAT ARE CLOSE TO THEIR MEAN

OLEKSIY KLURMAN, ALEXANDER P. MANGEREL, COSMIN POHOATA, AND JONI TERÄVÄINEN

Abstract. We introduce a simple approach to study partial sums of multiplicative functions which are close to their mean value. As a first application, we show that a completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies

$$\sum_{n \le x} f(n) = cx + O(1)$$

with $c \neq 0$ if and only if $f(p) = 1$ for all but finitely many primes and $|f(p)| < 1$ for the remaining primes. This answers a question of Imre Ruzsa.

For the case $c = 0$, we show, under the additional hypothesis $\sum_{p : |f(p)|<1} 1/p < \infty$, that f has bounded partial sums if and only if $f(p) = \chi(p)p^{it}$ for some non-principal Dirichlet character χ modulo q and $t \in \mathbb{R}$ except on a finite set of primes that contains the primes dividing q, wherein $|f(p)| < 1$. This essentially resolves another problem of Ruzsa and generalizes previous work of the first and the second author on Chudakov's conjecture.

We also consider some other applications, which include a proof of a recent conjecture of Aymone concerning the discrepancy of square-free supported multiplicative functions.

1. Introduction

Let S^1 denote the unit circle of the complex plane. The characterization of the behaviour of partial sums of multiplicative functions is an active object of study in analytic number theory, being an example of a problem in which the multiplicative structure of an additively structured set, namely an interval, is investigated. Considerations involving the extremal behaviour of such partial sums, for example characterizing when these are bounded, was of essential importance in Tao's solution in [13] to the Erdős Discrepancy Problem (EDP), according to which

$$\sup_{d,N \ge 1} \left| \sum_{n \le N} f(dn) \right| = \infty,$$

for any sequence $f : \mathbb{N} \to S^1$. It is not difficult to see that the completely multiplicative functions, which are insensitive to dilations by d, ought to be extremal for this problem, and indeed Tao's proof proceeds by reducing the problem to the case of completely multiplicative functions and then proving the claim in this case.

Chudakov [4] considered the problem of characterizing completely multiplicative functions with arithmetic constraints whose partial sums behave rigidly, specifically being very close to their mean value. He conjectured in particular that if $f : \mathbb{N} \to \mathbb{C}$ is a completely multiplicative function taking only finitely many values, vanishing at only finitely many primes, and satisfying

$$\sum_{n \le x} f(n) = cx + O(1) \tag{1}$$

for some $c \in \mathbb{C}$, then f is a Dirichlet character. We will say that such functions are *close to their mean.*

When $c \neq 0$ this was proven by Glazkov [5] by an intricate analytic method; the case $c = 0$ was proven by the first two authors in [11].

It is natural to ask if one can relax certain of the hypotheses in Chudakov's problem and still obtain a characterization of those functions that satisfy (1). Ruzsa posed the following question in this direction (see [3, Problem 21]).

Question 1.1 (Ruzsa). *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function that satisfies*

$$\sum_{n \le x} f(n) = cx + O(1).$$

as $x \to \infty$ for some $c \neq 0$. Is it true that for every prime p either $|f(p)| < 1$ or $f(p) = 1$ holds?

In Chudakov's problem, the finite range of f implies that f either vanishes or takes values on S^1 in roots of unity of bounded order. The main complication in 1.1 arises from the lack of any restriction on the possible values of $f(p)$ on S^1.

We answer Question 1.1 in the affirmative, in fact in a stronger form that gives a complete characterization of completely multiplicative functions that are close to their mean, provided the mean is non-zero.

Theorem 1.2. *A completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies*

$$\sum_{n \le x} f(n) = cx + O(1) \tag{2}$$

as $x \to \infty$ for some $c \neq 0$ if and only if there exists a finite set S of primes such that $|f(p)| < 1$ for $p \in S$ and $f(p) = 1$ for $p \notin S$.

Remark 1.1. It is easy to check that the above characterization fails when f is merely assumed multiplicative. We can construct many multiplicative functions f that are not completely multiplicative and that are close their mean as follows.

Let $h : \mathbb{N} \to \mathbb{C}$ be a multiplicative function satisfying

$$\sum_{d > y} |h(d)| \ll y^{-1} \tag{3}$$

for all y large. Let g be any completely multiplicative function satisfying (2). Set $f := g * h$. Then

$$\sum_{n \le x} f(n) = \sum_{d \le x} h(d) \sum_{m \le x/d} g(m) = cx \sum_{d \le x} \frac{h(d)}{d} + O(\sum_{d \le x} |h(d)|)$$
$$= cx \left(\sum_{d=1}^{\infty} \frac{h(d)}{d} + O(1/x) \right) + O(1) = c'x + O(1),$$

where c' is nonzero as long as $\sum_d \frac{h(d)}{d} \neq 0$. As an example, take g to be the constant function 1, and let $h(n) := \mu^2(n)/n^2$. Defining $f := g * h$, we have $f(p^k) = 1 + 1/p^2$ for all $k \ge 1$. This of course is not a completely multiplicative function, however.

In the same paper, Ruzsa also asked for a plausible characterization of completely multiplicative functions with bounded partial sums, i.e., the case $c = 0$ of Question 1.1 (see Problem 22 of [3]). This problem is more difficult than that of Chudakov, due to the possibility that the set of primes where f either vanishes or is quite small can be quite large. Nevertheless, we make some progress in this direction by finding a characterization of such completely multiplicative functions when we restrict the set where $|f(p)|$ is small.

Definition 1.1. Let S be a set of primes. We say that S is *thin* if $\sum_{p \in S} 1/p < \infty$.

Theorem 1.3. *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function, such that $\{p : |f(p)| < 1\}$ is thin. Then f satisfies*

$$\limsup_{x\to\infty} \left|\sum_{n\leq x} f(n)\right| < \infty$$

if and only if there is a primitive Dirichlet character χ, a real number t and a finite set of primes S (including those primes that divide the conductor of χ) such that $f(p) = \chi(p)p^{it}$ if $p \notin S$, and $|f(p)| < 1$ if $p \in S$.

Theorems 1.2 and 1.3 allow us to deduce a refinement of Chudakov's conjecture, whether or not $c \neq 0$. In particular, the finiteness of values can be relaxed in the case $c = 0$.

Corollary 1.4 (Refinement of Chudakov's Conjecture)**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function satisfying* (2).
a) If $c \neq 0$ and $|f(n)| \in \{0, 1\}$ for all n, then there is an integer $q \in \mathbb{N}$ such that f is the principal Dirichlet character modulo q.
b) If $c = 0$, and $|f(n)| \in \{0, 1\}$ for all n, and additionally $f(p) = 0$ for only finitely many primes, then there is a $t \in \mathbb{R}$ and a $q \in \mathbb{N}$ such that $f(n) = \chi(n)n^{it}$ for some Dirichlet character χ modulo q.

Completely removing an assumption like $\sum_{p:|f(p)|<1} 1/p < \infty$ in Theorem 1.3 seems to be very hard. Indeed, a crucial part of the necessity direction in our argument involves showing that a completely multiplicative function with bounded partial sums is pretentious (see Lemma 4.2 below). This type of reduction was also crucial in Tao's solution to the EDP.

In order to make this reduction, Tao showed that there was a non-zero shift $h \in \mathbb{Z}$ such that $|\sum_{n\leq x} f(n)\overline{f}(n+h)/n| \gg \log x$ at some large scale x, at which point his groundbreaking work on the logarithmically averaged Elliott conjecture [14] affirms that f is pretentious. The key fact that is used to demonstrate the existence of such a large logarithmic correlation is that

$$\sum_{n\leq x} |f(n)|^2/n \gg \log x, \tag{4}$$

a triviality when f takes values in S^1, but an assertion that is no longer guaranteed for more general functions.

It is a simple consequence of a result of Delange that if $\sum_p (1 - |f(p)|^2)/p = \infty$ then (4) fails, and we can no longer rely on this important step of reducing the problem to correlations of multiplicative functions. This is only mildly less restrictive than what we assume above (and is equivalent to the same condition when f takes its unimodular values in roots of unity of fixed order, say).

In Section 4 we provide some examples of completely multiplicative functions that fail to satisfy the supplementary condition, but nevertheless have bounded partial sums.

In Section 5 we address two more applications of the techniques used in this paper. One of these concerns a problem treated in a recent paper of Aymone [1] on the Erdős discrepancy problem for square-free supported multiplicative sequences. In [1], it is proven among other things, that if for a completely multiplicative function $g : \mathbb{N} \to \{-1, +1\}$ we have

$$\sum_{n\leq x} g(n)\mu^2(n) = O(1),$$

then there exists a real primitive Dirichlet character χ of conductor q such that $(q,6)=1$ and $g(2)\chi(2)=g(3)\chi(3)=-1$, and moreover,

$$\sum_p \frac{1-g(p)\chi(p)}{p}<\infty. \tag{5}$$

He conjectured (see Conjecture 1.1 of [1]) that no such functions g exist. The arguments of the present paper allow us to settle this conjecture.

Theorem 1.5. *Let $g:\mathbb{N}\to\{-1,+1\}$ be a multiplicative function. Then*

$$\limsup_{x\to\infty}\left|\sum_{n\le x} g(n)\mu^2(n)\right|=\infty.$$

1.1. **Proof ideas.** The proofs of all of the results mentioned above crucially use a variant of an idea (which we call the "rotation trick") that originated in the forthcoming work of the first, second and fourth authors on the Erdős discrepancy problem over function fields [10]. Roughly speaking, starting from the work of Tao [13] and subsequently in [8], [11], [9] after using Tao's theorem [14] to reduce to f being pretentious in the sense that $\mathbb{D}(f,\chi(n)n^{it},\infty)<\infty$, the authors used variants of the second moment argument to show that for $f:\mathbb{N}\to S^1$ multiplicative,

$$\frac{1}{x}\sum_{n\le x}\Big|\sum_{n\le m\le n+H} f(m)\Big|^2 \tag{6}$$

is large for appropriately chosen H unless f is of a very special form. The largeness of this mean square is demonstrated by expanding out the square and expressing the resulting expression as a linear combination of various correlations

$$\sum_{n\le x} f(n)\overline{f}(n+h),$$

for which we have explicit formulas in the pretentious case by [8].

This approach does not directly work in our applications, so we need a new approach in the pretentious case. Let us outline the proof of Theorem 1.3. As in [11], we can analyze (6) to conclude that if the partial sums of f are bounded, then $f(p)=\chi(p)p^{it}$ whenever $|f(p)|=1$. However, for primes p with $|f(p)|<1$ (of which there could be infinitely many, unlike in the proof of Chudakov's conjecture in [11]), this argument does not work and we must find a different approach, which involves the "rotation trick" mentioned earlier.

Here we describe a somewhat simplified version of the "rotation trick." Assuming for the sake of contradiction that there exist infinitely many primes p for which $f(p)\ne\chi(p)p^{it}$ (and assuming $t=0$ for simplicity), we use these primes via the lower bound sieve and the Chinese remainder theorem to construct two short intervals I and I' of length H such that

$$\left|\sum_{n\in I} f(n)-\sum_{n\in I'} f(n)\right| \tag{7}$$

is large; this in particular implies that f must exhibit large partial sums, which leads to a contradiction. The construction of these two intervals proceeds roughly as follows. Since the set $\mathcal{S}:=\{p:\ |f(p)|<1\}$ is thin by assumption, it is a zero-dimensional set from the point of view of sieve theory, and hence sieving works extremely well for this set. We can thus find many integers m for which all the numbers in $[(H!)^2m,(H!)^2+m+H]$ have no prime factors from $\mathcal{S}\cap[H+1,\infty)$. Letting $p_1,\dots,p_l>H$ be distinct primes from $\mathcal{S}$ and $r_1,\dots,r_l\in[1,H]$, $k_1,\dots,k_l\in\mathbb{N}$ be parameters, we can similarly find many m' such that for all $1\le j\le l$ we have $p_j^{k_j}\ ||\ (H!)^2m'+r_j$, and such that for all $1\le r\le H$ the condition $p\mid (H!)^2m'+r$,

$p \in \mathcal{S} \cap [H+1, \infty)$ implies $r = r_j$ and $p = p_j$ for some j. Now, it is not difficult to see that for $1 \le r \le H$, $r \neq r_j$, the numbers $(H!)^2 m + r$ and $(H!)^2 m' + r$ have the same prime factors from $\mathcal{S}$ with the same multiplicities, and thus

$$f((H!)^2 m + r) = f((H!)^2 m' + r),$$

since for $p \notin \mathcal{S}$ we already know from the arguments above that $f(p) = \chi(p)$. On the other hand, we similarly see that for $r = r_j$ the numbers $(H!)^2 m + r$ and $(H!)^2 m' + r$ have the same prime factors from $\mathcal{S}$ with the same multiplicities, apart from the prime p_j that only divides the latter number and divides it to power k_j. Thus

$$f((H!)^2 m + r) = (f(p_j)\overline{\chi}(p))^{k_j} f((H!)^2 m' + r), \quad r = r_j.$$

Now, taking $I = [(H!)^2 m, (H!)^2 + H]$, $I' = [(H!)^2 m', (H!)^2 m' + H]$, (7) becomes

$$\left| \sum_{1 \le j \le l} (1 - (f(p)\overline{\chi}(p))^{k_j}) f((H!)^2 m + r_j) \right|, \tag{8}$$

and this can easily be made large by choosing the k_j large (so that $|f(p_j^{k_j})| \le 1/2$, say) and by choosing the r_j appropriately so that the corresponding values of f all point in approximately the same direction. We call the underlying idea a "rotation trick", since effectively we have used a small set of primes to rotate the terms in the sums over $n \in I$ and $n \in I'$ to point in opposite directions.

The proof of Theorem 1.2 does not require the use of Tao's result but proceeds in a similar way after reducing to the pretentious case. In this case, we use the primes $f(p) \neq 1$ to "rotate" the sums in a suitable direction. In the proof of Theorem 1.5, we also follow similar ideas as above, but establish the largeness of (8) for suitable k_j and r_j in a slightly different manner. The outcome of this analysis is that if a function $g : \mathbb{N} \to \{-1, +1\}$ violates Theorem 1.5, then there must be a real Dirichlet character χ such that $g(p) = \chi(p)$ for all but finitely many p. For such g then, the above arguments are not applicable, but we instead apply a Dirichlet series argument that involves bounding the number of common zeros of $L(s, \chi)$ and $\zeta(2s)$; see Section 5 for the details.

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1.2. **Notation.** We denote by S^1 and $\mathbb{U}$ the unit circle and the closed unit disc of the complex plane, respectively. By $v_p(n)$ we denote the largest k such that $p^k \mid n$, and let $\mathrm{rad}(n)$ denote the product $\prod_{p|n} p$. The notation (a, b) stands for the greatest common divisor of a and b, and more generally, if $\mathcal{S} \subset \mathbb{N}$ is a set, then $(n, \mathcal{S}) := \max_{s \in \mathcal{S}} (n, s)$.

2. Proof of Theorem 1.2

2.1. **The if part.** This part is simple. Suppose $f(p) = 1$ for all primes except $p_1, \ldots, p_k$. Then for some complex numbers z_i with $|z_i| < 1$ we may write

$$f(n) = z_1^{v_{p_1}(n)} \cdots z_k^{v_{p_k}(n)}.$$

By the sieve of Eratosthenes, we have an exact formula for the summatory function as

$$\begin{aligned}\sum_{n \le x} f(n) &= \sum_{m_1, \ldots m_k \ge 0} z_1^{m_1} \cdots z_k^{m_k} \sum_{\substack{m \le x \\ p_i^{m_i} || m \ \forall i \le k}} 1 = \sum_{m_1, \ldots, m_k \ge 0} z_1^{m_1} \cdots z_k^{m_k} \sum_{r \le x/(p_1^{m_1} \cdots p_k^{m_k})} 1_{(r, p_1 \cdots p_k) = 1} \\ &= \sum_{m_1, \ldots m_k \ge 0} z_1^{m_1} \cdots z_k^{m_k} \sum_{d | p_1 \cdots p_k} \mu(d) \left\lfloor \frac{x}{p_1^{m_1} \cdots p_k^{m_k} d} \right\rfloor.\end{aligned}$$

Since

$$\sum_{m_1,\dots,m_k\geq 0}|z_1|^{m_1}\cdots|z_k|^{m_k}$$

converges, the sum over d above is up to $O(1)$ error equal to

$$\begin{aligned}\frac{x}{p_1^{m_1}\cdots p_k^{m_k}}\sum_{d|p_1\cdots p_k}\frac{\mu(d)}{d}&=\frac{x}{p_1^{m_1}\cdots p_k^{m_k}}\prod_{p|p_1\cdots p_k}\Big(1-\frac{1}{p}\Big)\\&:=c'\frac{x}{p_1^{m_1}\cdots p_k^{m_k}}\end{aligned}$$

for some $c'\neq 0$. Thus we conclude that

$$\sum_{n\leq x}f(n)=c'x\sum_{m_1,\dots m_k\geq 0}z_1^{m_1}\cdots z_k^{m_k}p_1^{-m_1}\cdots p_k^{-m_k}+O(1)=cx+O(1),$$

where, by the geometric sum formula,

$$c:=c'\prod_{1\leq i\leq k}\frac{1}{1-z_i/p_i}\neq 0.$$

The if part has now been proven.

2.2. **The only if part.** We begin with a few lemmas. The first lemma we need states the well-known theorems of Halász and Delange on mean values of multiplicative functions.

Lemma 2.1. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function. If the sum*

$$\sum_p\frac{1-\mathrm{Re}(f(p)p^{-it})}{p}$$

converges for some $t\in\mathbb{R}$, then

$$\frac{1}{x}\sum_{n\leq x}f(n)=\frac{x^{it}}{1+it}\prod_{p\leq x}\Big(1-\frac{1}{p}\Big)\Big(1+\sum_{k\geq 1}\frac{f(p^k)p^{-ikt}}{p^k}\Big)+o(1).$$

Otherwise, we have

$$\frac{1}{x}\sum_{n\leq x}f(n)=o(1).$$

Proof. See [15, Theorem III.4.5]. □

Definition 2.1. We say that a multiplicative function $f:\mathbb{N}\to\mathbb{C}$ has *property R* if it satisfies (2) for some $c\neq 0$.

Lemma 2.2. *Suppose that the completely multiplicative function $f:\mathbb{N}\to\mathbb{C}$ has property R. Then we have $|f(p)|\leq 1$ for all primes p.*

Proof. Suppose on the contrary that $|f(p_0)|>1$ for some prime p_0. Then by property R we have

$$f(p_0^k)=\sum_{n\leq p_0^k}f(n)-\sum_{n\leq p_0^k-1}f(n)=O(1),$$

but on the other hand the sequence $f(p_0^k)=f(p_0)^k$ tends to infinity in absolute value as $k\to\infty$, a contradiction. □

Lemma 2.3. *Suppose that the completely multiplicative function $f:\mathbb{N}\to\mathbb{C}$ has property R. Then $|f(p)|=1$ for all but finitely many primes p.*

Proof. Let $c \neq 0$ be the constant for which f satisfies property R. Suppose on the contrary that $p_1 < p_2 < \dots$ is an infinite sequence of primes with $|f(p_i)| < 1$. Choose $H \geq 1$ large. For each i, pick $m_i \in \mathbb{N}$ such that $|f(p_i^{m_i})| = |f(p_i)|^{m_i} \leq |c|/2$. Applying the Chinese remainder theorem, we can find an integer $n = n_H \geq 1$ such that $p_i^{m_i} \mid n_H + i$ for each $1 \leq i \leq H$. Then, employing Lemma 2.2, we have

$$|c|H + O(1) = \Big| \sum_{i=1}^{H} f(n_H + i) \Big| \leq \sum_{i \leq H} |f(p_i^{m_i})| \leq \frac{|c|}{2} H,$$

and as $H \to \infty$ this leads to a contradiction. □

Lemma 2.4. *Suppose that the completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$ has property R. Then both the sum $\sum_p \frac{1-\mathrm{Re}(f(p))}{p}$ and the product*

$$\prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p} + \frac{f(p)^2}{p^2} + \cdots\Big) \tag{9}$$

converge.

Proof. Since we have $|\sum_{n \leq x} f(n)| \gg x$, by Lemmas 2.2 and 2.1 there must exist some real number t such that

$$\sum_p \frac{1 - \mathrm{Re}(f(p)p^{-it})}{p} < \infty. \tag{10}$$

We suppose for the sake of contradiction that $t \neq 0$. Lemma 2.1 then gives the asymptotic

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{x^{it}}{1 + it} \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p)^2}{p^{2+2it}} + \cdots\Big) + o(1) := c' x^{it} + o(1)$$

for some constant c', since the product over p above converges by (10). But since f satisfies property R with some constant c, this implies

$$c = c' x^{it} + o(1),$$

as $x \to \infty$, which is an evident contradiction since $m \mapsto m^{it}$, $m \geq x_0$ is dense on the unit circle. Thus (10) holds with $t = 0$, and comparing sums and products we also obtain (9). □

Lemma 2.5. *Let $a(j)$ be complex numbers such that $a(j) \in S^1 \setminus \{1\}$ for all but finitely many j. Then we can find a complex number $w \in S^1$ with $|w - 1| \geq 0.1$, a sequence $\mathcal{J} \subseteq \mathbb{N}$ and numbers $k_j \in \mathbb{N}$ such that $a(j)^{k_j} \xrightarrow{j \to \infty, j \in \mathcal{J}} w$.*

Proof. Since $a(j) \in S^1 \setminus \{1\}$ for all large j, for such j we can pick natural numbers m_j so that $|a(j)^{m_j} - 1| \geq 0.1$ (if $a(j)$ is a root of unity of some order, then this is evidently possible by the periodicity of the powers; otherwise if $a(j) = e(\gamma_j)$ with γ_j irrational, then this is possible since $(\gamma_j n \pmod 1)_{n \geq 1}$ is dense on $[0, 1]$). The sequence $a(j)^{m_j}$ is a bounded sequence of complex numbers, so by compactness we can find a convergent subsequence that converges to some $w \in S^1$. By the condition on m_j, we must have $|w - 1| \geq 0.1$. □

Now, let H be an integer parameter that we will eventually send to infinity. Let M be large enough in terms of H; in particular, choose M so that

$$\Big| \sum_{p \geq M} \frac{1 - \mathrm{Re}(f(p))}{p} \Big| \leq 1/H^2.$$

Let $S := \{p : f(p) \neq 1\}$. We may assume that $|S| = \infty$, since otherwise (in view of Lemma 2.2) there is nothing to prove. Applying Lemmas 2.5 and 2.3, we find some $w \in S^1$ with $|w - 1| \geq 0.1$ and exponents $\alpha_p \in \mathbb{N}$ such that $f(p)^{\alpha_p} \xrightarrow{p \to \infty, p \in S'} w$, where the limit is over some infinite subset $S' \subseteq S$. We may assume that $p > M$ for $p \in S'$. Without loss of generality, we may also choose M so large that $|f(p)^{\alpha_p} - w| \leq 1/H^2$ for all $p \in S'$.

Next, with the sequence $p_1, p_2, \ldots$ of primes in S' and $w \in S^1 \setminus \{1\}$ as above, we apply the Chinese remainder theorem to find an integer $r \ll_M 1$ such that

$$\begin{aligned} &r \equiv 0 \pmod{(M!)^2}, \\ &p_i^{\alpha_i} \mid\mid r+i \quad \text{for all} \quad 1 \leq i \leq H. \end{aligned}$$

Set

$$Q := \prod_{i \leq H} p_i^{\alpha_i+1} \prod_{p \leq M} p^{M+1}, \quad d_i := (Q, r+i), \quad Q_i := Q/d_i. \tag{11}$$

Note that we have

$$\left(\frac{Q}{d_i}, \frac{r+i}{d_i}\right) = 1. \tag{12}$$

We consider the double sum

$$\Sigma := \sum_{i \leq H} \sum_{n \leq x} f(Qn + r + i).$$

On the one hand, changing the order of summation and applying the fact that f has property R, we see that

$$\Sigma = (cH + O(1))x. \tag{13}$$

On the other hand, by (11) we have

$$\Sigma = \sum_{i \leq H} f(d_i) \sum_{n \leq x} f\Big(Q_i n + \frac{r+i}{d_i}\Big).$$

By our conditions on r, for $p \leq M$ we have $f(p^{v_p((r+i,Q))}) = f(p^{v_p(r+i)}) = f(p^{v_p(i)})$ and $f(p_i^{\alpha_i}) = f(p_i)^{\alpha_i} = w + O(1/H^2)$f for $i \leq H$, so

$$f(d_i) = \prod_{p | d_i} f(p^{v_p(d_i)}) = (w + O(1/H^2))f(i),$$

since the prime factors of $d_i = (r+i, Q)$ are precisely the primes p dividing i and the prime p_i, since $p_j \nmid r+i$ for $i \neq j$ by the fact that $|i-j| < M < p_j$.

By making the change of variables $m = Q_i n + (r+i)/d_i$ and expanding the indicator $1_{m \equiv (r+i)/d_i \pmod{Q_i}}$ using the orthogonality of characters, formula (12) and Lemma 2.1 implies that

$$\begin{aligned} \sum_{n \leq x} f\Big(Q_i n + \frac{r+i}{d_i}\Big) &= \frac{1}{\phi(Q_i)}\Big(\sum_{m \leq Q_i x} f(m) 1_{(m,Q_i)=1} + \sum_{\substack{\chi \pmod{Q_i} \\ \chi \neq \chi_0}} \chi(\frac{r+i}{d_i}) \sum_{m \leq Q_i x} f(m)\overline{\chi}(m)\Big) + o(x) \\ &= \frac{1}{\phi(Q_i)} \sum_{m \leq Q_i x} f(m) 1_{(m,Q_i)=1} + o(x), \end{aligned} \tag{14}$$

since from Lemma 2.4 and the triangle inequality for the pretentious distance and the Vinogradov–Korobov zero-free region for Dirichlet L-functions it immediately follows that $\sum_p \frac{1-\mathrm{Re}(f(p)\overline{\chi}(p)p^{-it})}{p} = \infty$ for all non principal χ and any fixed $t \in \mathbb{R}$ (this is also implied, e.g., by Theorem 2 of [2]).

Further, by Lemma 2.1 and the fact that $p \mid Q_i$ for all $p \leq M$, for any $1 \leq i \leq H$ the expression (14) is

$$\begin{aligned}
&\frac{1}{\phi(Q_i)} Q_i x \prod_{p \mid Q_i} \Big(1 - \frac{1}{p}\Big) \prod_{\substack{p \leq x \\ p \nmid Q_i}} \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\Big) + o(x) \\
&= (1 + o(1)) x \Big(1 + O\Big(\sum_{p > M} \frac{1 - \mathrm{Re}(f(p))}{p}\Big)\Big) \\
&= (1 + O(1/H^2)) x.
\end{aligned}$$

Combining the contributions of different i gives

$$\Sigma = \sum_{i \leq H} (w + O(1/H^2)) f(i) \cdot x$$

However, by property R again and the fact that $|f(i)| \leq 1$ for all i, this yields

$$\Sigma = (cwH + O(1))x$$

so comparing with (13) we reach

$$c = cw + O(1/H).$$

Finally, letting $H \to \infty$ and using the facts that $w \neq 1$ and $c \neq 0$, we obtain the desired contradiction from this. Thus we must have $f(p) = 1$ for all but finitely many primes p, and now the only if direction of Theorem 1.2 has also been proven.

Proof of Corollary 1.4 a). Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function which satisfies (2) and $|f(n)| \in \{0, 1\}$ for all n. By Theorem 1.2 there is a finite set of primes S such that for all $p \notin S$ we have $f(p) = 1$, and otherwise $|f(p)| < 1$, so actually $f(p) = 0$.

Set now $q := \prod_{p \in S} p$. If $(m, q) = 1$ then f is identically 1 on all the primes that divide m, and hence $f(m) = 1$; if $(m, q) > 1$ then there is a prime dividing m where $f(p) = 0$, so by multiplicativity we have $f(m) = 0$. Hence $f(m) = 1_{(m,q)=1}$, or equivalently, f is the principal character modulo q. □

3. The case of bounded partial sums

In this section we prove Theorem 1.3. In the previous section, a key reduction step involved showing that f *pretended* to be the function 1, in the sense that $f(p) \approx 1$ for all but a thin set of primes p. With this reduction in place, we then used the Chinese remainder theorem to find many short intervals for which the corresponding partial sums were pointing in the same direction, and we used this to deduce that the thin set was in fact finite.

In this section we follow a similar strategy. Beginning with a completely multiplicative with bounded partial sums, we will shortly reduce to a case in which $f(p) \approx \chi(p)p^{it}$ for some $t \in \mathbb{R}$ and χ a primitive Dirichlet character of bounded conductor, outside of a thin set of exceptions. Our task will then be to show that this exceptional set is finite by a variant of the double sum argument above.

To further demonstrate the parallels, we begin with a definition.

Definition 3.1. We say that a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ has *property B* if

$$\limsup_{x \to \infty} \Big| \sum_{n \leq x} f(n) \Big| < \infty.$$

Lemma 3.1. *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function that satisfies property B. Then $|f| \leq 1$.*

Proof. This is the same as the proof of Lemma 2.2. □

Lemma 3.2. *Suppose that $f : \mathbb{N} \to \mathbb{U}$ satisfies property B, such that $\{p : |f(p)| < 1\}$ is thin in the sense of Definition 1.1. Then there is a primitive non-principal Dirichlet character χ and a real number t such that*

$$\sum_p \frac{1 - \mathrm{Re}(f(p)\overline{\chi}(p)p^{-it})}{p} < \infty.$$

Proof. Observe that

$$\sum_p \frac{1 - |f(p)|^2}{p} = \sum_{p:|f(p)|<1} \frac{1 - |f(p)|^2}{p} \le \sum_{p:|f(p)|<1} \frac{1}{p} < \infty.$$

By Lemma 2.1, we thus see that

$$\sum_{n \le x} |f(n)|^2 \gg x, \tag{15}$$

and thus by partial summation

$$\sum_{n \le x} \frac{|f(n)|^2}{n} \gg \log x.$$

We now use the second moment method as in the work of Tao [13]. Given $H \ge 1$ and x chosen sufficiently large in terms of H,

$$\begin{aligned}
1 &\gg \frac{1}{\log x} \sum_{n \le x} \frac{1}{n} \left| \sum_{n \le m \le n+H} f(n) \right|^2 \\
&= \frac{1}{\log x} \sum_{|h| \le H} \sum_{\max\{1,1-h\} \le m \le \min\{x,x+h\}} f(m)\overline{f}(m+h) \sum_{\max\{m-H,m+h-H\} \le n \le \min\{m,m+h\}} \frac{1}{n} \\
&\ge \frac{H}{\log x} \sum_{n \le x} \frac{|f(n)|^2}{n} - \frac{1}{\log x} \sum_{1 \le |h| \le H} (H + 1 - |h|) \left| \sum_{1 \le n \le x} \frac{f(n)\overline{f}(n+h)}{n} \right| - o(1),
\end{aligned}$$

and therefore also that

$$\max_{1 \le |h| \le H} \left| \sum_{1 \le n \le x} \frac{f(n)\overline{f}(n+h)}{n} \right| \ge \frac{1}{3H} \sum_{n \le x} \frac{|f(n)|^2}{n} - o(1) \gg_H \log x.$$

Combining Tao's theorem ([14, Theorem 1.3]) and Elliott's lemma (see [11, Lemma 2.3]), we can find a primitive character χ of conductor $q = O(1)$ and a real $t = O(1)$, both independent of x, such that

$$\sum_p \frac{1 - \mathrm{Re}(f(p)\overline{\chi}(p)p^{-it})}{p} < \infty.$$

It remains to check that χ is non-principal. If χ were principal then by Lemma 2.1 we would have

$$\left| \sum_{n \le x} f(n) \right| = (1 + o(1)) \frac{x}{|1 + it|} \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left|1 - \frac{f(p)p^{-it}}{p}\right|^{-1} \gg_t x \exp\left(-\sum_{p \le x} \frac{1 - \mathrm{Re}(f(p)p^{-it})}{p}\right) \gg x,$$

a contradiction with the boundedness of the partial sums when x is sufficiently large. Thus χ must be non-principal. □

When the conclusion of Lemma 3.2 holds for a given multiplicative function f, we say that f is *pretentious*, and that it *pretends to be* the twisted Dirichlet character $\chi(n)n^{it}$. In the sequel, it will be convenient to be able to dispose of the twist n^{it}. For this purpose, we introduce the following additional definition.

Definition 3.2. Let $f : \mathbb{N} \to \mathbb{U}$ be completely multiplicative. We say that f satisfies *Property B'* if

$$\limsup_{x\to\infty} \max_{1\le z\le x} \left|\sum_{x<n\le x+z} f(n)\right| < \infty.$$

Obviously any function satisfying Property B also satisfies Property B' by the triangle inequality. Importantly, twists of such functions by characters n^{it} with $t \in \mathbb{R}$ also satisfy Property B'.

Lemma 3.3. *Let $x \ge 2$. Let $f : \mathbb{N} \to \mathbb{U}$ be a completely multiplicative function satisfying Property B. Then for any fixed $t \in \mathbb{R}$ the function $f(n)n^{-it}$ satisfies Property B'.*

Proof. Fix $t \in \mathbb{R}$, and let $x \ge 2$. Choose $z \in [1, x]$ that maximizes $y \mapsto \left|\sum_{x<n\le x+y} f(n)n^{-it}\right|$ with $y = z$. If $t = 0$ then the boundedness follows from the triangle inequality and Property B, so suppose $t \ne 0$. In this case we apply partial summation as

$$\sum_{x<n\le x+z} f(n)n^{-it} = (x+z)^{-it} \sum_{n\le x+z} f(n) - x^{-it} \sum_{n\le x} f(n) + it \int_x^{x+z} \left(\sum_{n\le u} f(n)\right) \frac{du}{u^{1+it}}.$$

Taking absolute values and applying the triangle inequality and Property B, we conclude that

$$\left|\sum_{x<n\le x+z} f(n)n^{-it}\right| \ll \left(\max_{x'\in[x,2x]} \left|\sum_{n\le x'} f(n)\right|\right) \left(1 + \int_x^{2x} \frac{du}{u}\right) \ll 1,$$

uniformly in x. The claim follows. □

Having concluded this, we work below with functions f, satisfying Property B', that pretend to be a genuine primitive non-principal Dirichlet character χ.

Proposition 3.4. *Let $f : \mathbb{N} \to \mathbb{U}$ be a completely multiplicative that pretends to be a primitive Dirichlet character χ of conductor $q > 1$. Assume furthermore that f satisfies Property B', and that $\{p : |f(p)| < 1\}$ is thin. Then the following hold:*
a) for all primes $p \nmid q$ where $|f(p)| = 1$ we have $f(p) = \chi(p)$;
b) for all primes $p|q$ we have $|f(p)| < 1$;
c) for every fixed $\eta > 0$ the set $\{p : |1 - f(p)\overline{\chi}(p)| \ge \eta\}$ is finite.

To prove this result, we shall rely on previous work of the first two authors on Chudakov's conjecture (see [11, Section 5]).

Proof. Before proceeding to the proof of the various parts of this proposition, we make some preliminary observations based on the work of [11].

First, if[1] $f(2) \ne 0$ we set $q' := q$ and define $\chi' := \chi$, while if $f(2) = 0$ then set $q' := [q, 2]$ and $\chi' := \chi\chi_0^{(2)}$, where $\chi_0^{(2)}$ is the principal character modulo 2. Then f pretends to be χ', and properties a), b) ,c) are unaffected by replacing χ by χ'. This will simplify our work in

[1]The same manoeuvre was employed in [11], so there is no loss in consistency at this step with what is done there.

what follows.
Since f satisfies Property B', for x large we have

$$\frac{1}{x}\sum_{n\leq x}\left|\sum_{n\leq m\leq n+H} f(n)\right|^2 = \frac{1}{x}\sum_{H<n\leq x}\left|\sum_{n\leq m\leq n+H} f(n)\right|^2 + O(H^3/x) \ll 1+H^3/x.$$

Expanding the expression on the left hand side, we find that

$$\begin{aligned}\frac{1}{x}\sum_{n\leq x}\left|\sum_{n\leq m\leq n+H} f(n)\right|^2 &= \frac{1}{x}\sum_{n\leq x}\sum_{n\leq m_1,m_2\leq n+H} f(m_1)\overline{f(m_2)}\\ &= \sum_{|h|\leq H}\frac{1}{x}\sum_{m\leq x+H} f(m)\overline{f}(m+h)\sum_{\max\{m-H,m+h-H\}\leq n\leq \min\{m,m+h\}} 1\\ &= \sum_{|h|\leq H}(H+1-|h|)\cdot\frac{1}{x}\sum_{m\leq x} f(m)\overline{f}(m+h) + O(H^3/x).\end{aligned}$$

Taking $x\to\infty$ and applying [8, Theorem 1.5] (in the form stated in (21) of [11]), we obtain

(16)
$$1 \gg \lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\left|\sum_{n\leq m\leq n+H} f(n)\right|^2 = \frac{1}{q}\sum_{d\geq 1}\frac{G(d)}{d}\sum_{\mathrm{rad}(R)|q'}\frac{|f(R)|^2}{R}\sum_{\substack{|h|\leq H\\ R|h, d|h/R}}(H+1-|h|)S_{\chi'}(|h|/R),$$

where, given $m\in\mathbb{Z}$ we have written

$$S_{\chi'}(m) := \sum_{a \pmod{q'}} \chi'(a)\overline{\chi'}(a+m),$$

and for each $d\geq 1$, the local correlation factor[2] $G(d)$ is defined as

$$G(d) := \prod_{\substack{p^k||d\\ k\geq 0}}\left(|\mu * F(p^k)|^2 + 2\mathrm{Re}\left(\sum_{i>k}\frac{\mu * F(p^i)\overline{\mu * F}(p^k)}{p^{i-k}}\right)\right),$$

with F being the completely multiplicative function given by $F(p) := f(p)\overline{\chi'}(p)$ if $p\nmid q'$ and $F(p)=1$ otherwise. Note that $F(2)\neq 0$ since either $2|q'$ in which case $F(2)=1$ automatically, or else $2\nmid q'$, in which case $f(2)\chi(2)\neq 0$.

A simple calculation of the $G(d)$ as a geometric sum shows that[3]

$$\begin{aligned}G(d) &= \prod_{p\nmid d}\left(1+2\mathrm{Re}\left(\frac{F(p)-1}{p-F(p)}\right)\right)\cdot\prod_{\substack{p^k||d\\ k\geq 1}}|F(p)|^{2(k-1)}|F(p)-1|^2\left(1+2\mathrm{Re}\left(\frac{F(p)}{p-F(p)}\right)\right)\\ &= G_1(d)G_3(d),\end{aligned}$$

[2]One may interpret $G(d)$ as the local correlation of F and $\overline{F}$ at the primes dividing d with shift d; compare [8, page 5].

[3]In the case that $p||d$ and $F(p)=0$, we use the convention $0^0=1$.

where we define

$$G_1(d) := \prod_{p \nmid 3d} \left(1 + 2\mathrm{Re}\left(\frac{F(p)-1}{p-F(p)}\right)\right) \cdot \prod_{\substack{p^k || d \\ p \neq 3}} |F(p)|^{2(k-1)} |F(p)-1|^2 \left(1 + 2\mathrm{Re}\left(\frac{F(p)}{p-F(p)}\right)\right)$$

$$G_3(d) := \begin{cases} |F(3)|^{2(k-1)}|F(3)-1|^2(1+2\mathrm{Re}(F(3)/(3-F(3)))) & \text{if } 3^k || d \\ 1 + 2\mathrm{Re}\left(\frac{F(3)-1}{3-F(3)}\right) \text{ if } 3 \nmid d. \end{cases}$$

We note for future reference that

$$G_3(d) \geq 0 \text{ with equality if, and only if, either } 3|d \text{ and } F(3) = 1, \text{ or } 3 \nmid d \text{ and } F(3) = -1.$$

We have $G_1(1) \neq 0$ since $1 + 2\mathrm{Re}((F(p)-1)/(p-F(p))) = 0$ iff $p = 2 - F(p)$, which is impossible for $p \neq 3$ (as $F(2) \neq 0$). Factoring out $G_1(1)$ from $G_1(d)$, we produce (by a similar calculation as in (20) of [11]) a multiplicative function

$$\begin{aligned} \tilde{G}_1(d) := G_1(d)/G_1(1) &= \prod_{\substack{p^k || d \\ p \neq 3}} |F(p)|^{2(k-1)} |F(p)-1|^2 \frac{p^2-1}{(p-1)^2 - 2(1-\mathrm{Re}(F(p)))} \\ &= \tilde{G}_{1,s}(d) \prod_{\substack{p^k || d \\ p \neq 3}} |F(p)|^{2(k-1)}, \end{aligned}$$

where $\tilde{G}_{1,s}$ is the strongly multiplicative[4] function defined on prime powers by

$$\tilde{G}_{1,s}(p^k) := \begin{cases} |F(p)-1|^2 \frac{p^2-1}{(p-1)^2-2(1-\mathrm{Re}(F(p)))} & \text{if } p \neq 3 \\ 1 & \text{if } p = 3 \end{cases}$$

for all $k \geq 1$. We then set $\tilde{G}(d) := \tilde{G}_1(d) G_3(d) = G(d)/G_1(1)$, which obeys the relations

$$\begin{aligned} \tilde{G}(3^b m) &= \tilde{G}_1(m) G_3(3^b) \text{ whenever } 3 \nmid m \text{ and } b \geq 0, \\ \tilde{G}(mn) &= \tilde{G}(m)\tilde{G}_1(n) \text{ whenever } m, n \in \mathbb{N}, 3 \nmid m. \end{aligned}$$

It is easily seen that $\tilde{G}(d) \geq 0$ whenever d is odd. Since $F(p) = 1$ for all $p|q'$ we also see that $\tilde{G}(d) = 0$ whenever $(d, q') > 1$. Furthermore, applying (16) with $H = 0$ (so that $h = 0$), we deduce that

$$\sum_{\substack{d \geq 1 \\ (d,q')=1}} \frac{G(d)}{d} = \frac{q'}{\phi(q')} \prod_{p|q'} \left(1 - \frac{|f(p)|^2}{p}\right) \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^2 > 0, \tag{17}$$

as in (15).

Our goal is to exploit (16) to show that $|f(p)| < 1$ for $p|q'$ in b) and $F(p) = 1$ for all $p \nmid q'$ in a), and a crucial step will be to show that the terms on the right of (16) are all non-negative. If $2|q'$ then $\tilde{G}(d) \geq 0$ for all $(d, q') = 1$, which will make the remaining arguments simpler. Thus, let us assume that $2 \nmid q'$.

To deal with the prime $p = 2$, for which we may potentially have $\tilde{G}(2^k) < 0$, we observe that

$$\sum_{\substack{d \geq 1 \\ (d,q')=1}} \frac{G(d)}{d} = G_1(1) \left(1 + \sum_{k \geq 1} \frac{\tilde{G}_1(2^k)}{2^k}\right) \sum_{\substack{d \geq 1 \\ (d,2q')=1}} \frac{\tilde{G}(d)}{d} = G_1(1) \left(1 + \frac{\tilde{G}_{1,s}(2)}{2 - |F(2)|^2}\right) \sum_{\substack{d \geq 1 \\ (d,2q')=1}} \frac{\tilde{G}(d)}{d}.$$

[4] We say that $G : \mathbb{N} \to \mathbb{C}$ is strongly multiplicative if G is multiplicative and additionally $G(p^k) = G(p)$ for all primes p and for $k \geq 1$.

By (17) the series on the left hand side is positive, and since $\tilde{G}(d) \geq 0$ for all odd d the series on the right hand side is also non-negative. It follows that

$$G_1(1) \left(1 + \frac{\tilde{G}_{1,s}(2)}{2 - |F(2)|^2}\right) > 0.$$

Since $G_1(d)$ is non-negative for odd $d \geq 3$, the sign of $G_1(1)$ is the same as that of $\tilde{G}_{1,s}(2)$ (both of which arise from the factor at $p = 2$). If $G_1(1) \geq 0$ then $\tilde{G}_{1,s}(2) \geq 0$ (and thus $\tilde{G}_1(2^k) \geq 0$ for all k) as well. Otherwise, $G_1(1) < 0$ and thus $\tilde{G}_{1,s}(2) < -2 + |F(2)|^2$. Following the proof of [11, Proposition 5.3] (but with $\eta := |F(2)|^2 + |\tilde{G}_{1,s}(2)| > 2$ in the case that $\tilde{G}_{1,s}(2) < 0$), we may conclude from (16) that

$$\sum_{\substack{d \geq 1 \\ (d, 2q')=1}} \tilde{G}(d) \sum_{\mathrm{rad}(R)|q'} |f(R)|^2 \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \frac{\mu(g)}{g^2} \left\| \frac{Hg}{dR} \right\| = O(1), \tag{18}$$

where $\|t\| := \min_{n \in \mathbb{Z}} |t - n|$.

Lemma 5.5 of [11] shows that for any $t \in \mathbb{R}$ and $m \in \mathbb{N}$ odd and squarefree,

$$\sum_{g|m} \mu(g) g^{-2} \|gt\| \geq 0.$$

Applying this in (18) (with $t = H/(dR)$, for all d and R in range), we conclude that all of the terms on the left hand side of (18) are non-negative. We shall base the proofs of a), b) and c) on (18) and the fact that its terms are all non-negative below.

We derive from this information that $F(3) \neq -1$. Indeed, suppose otherwise. Then $\tilde{G}(d) = 0$ for all $3 \nmid d$, as noted earlier. Let $M \geq 1$ be large. Then applying (18) and dropping all of the terms besides $R = 1$ and $d = 3^m$ for all $1 \leq m \leq M$, we obtain

$$\sum_{1 \leq m \leq M} \tilde{G}(3^m) \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \left\| \frac{Hg}{3^m} \right\| = O(1).$$

We select $H = 3^{M+1}/2$, so that for each $1 \leq m \leq M$ and $g|\mathrm{rad}(q'/2^{v_2(q')})$, the number $2Hg/3^m$ is an odd integer. Thus, $\|Hg/3^m\| = 1/2$ for each pair (m, g) in play, and since $\sum_{g|N} \mu(g) g^{-2} = \prod_{p|N}(1 - 1/p^2) \asymp 1$, we see that

$$\sum_{1 \leq m \leq M} \tilde{G}(3^m) = O(1).$$

But $\tilde{G}(3^m) = \tilde{G}_1(3^m) G_3(3^m) = 2$ when $F(3) = -1$, and hence $M = O(1)$, which is a contradiction since M can be as large as desired. Hence, in what follows we may assume that $F(3) \neq -1$.

a) Let $p \geq 3$ be a prime not dividing q' for which $|f(p)| = 1$, and let $M \geq 1$. Then $|F(p)| = 1$ as well; assume that $F(p) \neq 1$. Since $F(3) \neq -1$, we have $G_3(d) = G_3(1) > 0$ whenever $(d, 3) = 1$. Thus, if $p \geq 5$ then by positivity, we can drop all terms in (18) except $R = 1$ and $d = p^m$ for $m \leq M$ and divide through by $G_3(p^m) = G_3(1)$ to conclude that

$$\sum_{m \leq M} \tilde{G}_1(p^m) \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \|Hg/p^m\| = O(1).$$

Picking $H := \frac{1}{2} p^{M+1}$, we have $\|Hg/p^m\| = 1/2$ for all $g|\mathrm{rad}(q'/2^{v_2(q')})$ and all $m \leq M$, as in the previous paragraph. It follows again that

$$\sum_{m \leq M} \tilde{G}_1(p^m) = O(1).$$

As $|F(p)| = 1$ and $F(p) \neq 1$,

$$\tilde{G}_1(p^m) = \tilde{G}_{1,s}(p^m) = \tilde{G}_{1,s}(p) \gg |F(p)-1|^2 > 0$$

for all $1 \leq m \leq M$. It follows that $M|F(p)-1|^2 = O(1)$ for any $M \in \mathbb{N}$, which is impossible. Thus, if $p > 3$ does not divide q' and $|f(p)| = 1$ then $F(p) = 1$. In particular, if $p|d$ for any $p > 3$ then $G(d) = \tilde{G}(d) = 0$.

If $p = 3$ and $|F(3)| \neq 1$ there is nothing to prove here, so assume $|F(3)| = 1$, noting still that $F(3) \neq \pm 1$. Then we instead use the fact that $\tilde{G}_1(3^m) = 1$ for all m and $H = 3^{M+1}/2$, to obtain

$$\sum_{m \leq M} G_3(3^m) = O(1),$$

and as $|F(3)| = 1$ we also have $G_3(3^m) = G_3(3) \gg |F(3)-1|^2 > 0$, so the contradiction $G_3(3)M = O(1)$ arises with $p = 3$. Thus, $F(3) = 1$ as well.

It remains to check the case $p = 2$. If $2|q'$ or $|f(2)| < 1$ then we are done since these cases are excluded; otherwise q' is odd and $|F(2)| = |f(2)| = 1$, and we assume that $F(2) \neq 1$. Lemma 5.4 of [11], which follows from (16) (and whose proof uses nothing about G besides the property $G(d) = 0$ for $(d, q') > 1$), gives

$$\sum_{\substack{d \geq 1 \\ (d,q')=1}} \tilde{G}(d) \sum_{\mathrm{rad}(R)|q'} |f(R)|^2 \sum_{g|\mathrm{rad}(q')} \mu(g) g^{-2} \left\| \frac{Hg}{dR} \right\| = O(1). \tag{19}$$

As shown above, $\tilde{G}(p) = 0$ whenever $p \neq 2$, and since $\tilde{G}_1$ is multiplicative this implies that $\tilde{G}(d) = 0$ provided $d \neq 2^k$ for some k. The above sum thus simplifies to

$$|\tilde{G}_{1,s}(2)| \sum_{k \geq 1} |F(2)|^{2(k-1)} \sum_{\mathrm{rad}(R)|q'} |f(R)|^2 \sum_{g|\mathrm{rad}(q')} \mu(g) g^{-2} \|Hg/2^k R\| = O(1).$$

Let $K, J \geq 1$. By positivity we may restrict the sum in R to $R = 1$, and the sum in k to a set $\{k_j\}_{j \leq J}$, chosen as precisely those $k \leq K$ such that $2^{K-k} \equiv 1 \pmod{\mathrm{rad}(q')}$ (recall that q' is assumed odd here). As $|F(2)| = 1$ once again, we in fact have

$$|\tilde{G}_{1,s}(2)| \sum_{1 \leq j \leq J} \sum_{g|\mathrm{rad}(q')} \mu(g) g^{-2} \|Hg/2^{k_j}\| = O(1).$$

Select $H := 2^K/\mathrm{rad}(q')$. By choice of k_j, we have that $\|Hg/2^{k_j}\| = \|g/\mathrm{rad}(q')\| = g/\mathrm{rad}(q')$ as long as $g \neq \mathrm{rad}(q')$, and $\|H\mathrm{rad}(q')/2^{k_j}\| = 0$. It follows that for each $1 \leq j \leq J$ we obtain

$$\sum_{g|\mathrm{rad}(q')} \mu(g) g^{-2} \|Hg/2^{k_j}\| = \frac{1}{\mathrm{rad}(q')} \sum_{\substack{g|\mathrm{rad}(q') \\ g \neq \mathrm{rad}(q')}} \mu(g) g^{-1} = \frac{1}{\mathrm{rad}(q')} \left(\frac{\phi(q')}{q'} - \frac{\mu(\mathrm{rad}(q'))}{\mathrm{rad}(q')} \right) \neq 0.$$

It follows that $|\tilde{G}_{1,s}(2)| J \ll 1$, which is a contradiction since K, and thus J, can be arbitrarily large. Thus, $F(2) = 1$ as well.

To conclude, $f(p)\overline{\chi}(p) = F(p) = 1$ for all $p \nmid q'$ with $|f(p)| = 1$, which implies the claim.

b) Suppose for the sake of contradiction that we can find a prime p dividing q' with $|f(p)| = 1$. Let $M \geq 1$. If p is odd then, beginning once again with (18), we use positivity and drop all of the terms in d except for $d = 1$, and all of the terms in R aside from powers p^m with $m \leq M$. This gives

$$\sum_{m \leq M} |f(p)|^m \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \|Hg/p^m\| = \sum_{m \leq M} \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \|Hg/p^m\| = O(1),$$

uniformly over H. Setting $H = p^{M+1}/2$ as in a), we have $\|Hg/p^m\| = 1/2$ for all $m \leq M$. It follows that $M = O(1)$, which is a contradiction since M can be taken arbitrarily large.

Thus, $|f(p)| < 1$ for all $p|q'$ odd. If $p = 2$ divides q' then the same contradiction arises from restricting the sum in (19) to $R = 2^k$, and arguing as in part a).

c) Let $\eta > 0$. Define $\mathcal{E}_\eta := \{p \geq 5 : |1 - F(p)| \geq \eta\}$. Since the set of primes dividing q', which is the collection of primes where $F(p) \neq f(p)\overline{\chi'}(p)$, is finite, to complete the proof of c) it suffices to check that $|\mathcal{E}_\eta| < \infty$.

Assume otherwise. Let M be chosen large in terms of η. Let $p_1, \ldots, p_M \in \mathcal{E}_\eta$ be distinct, necessarily odd, primes. We restrict the sum in (18) in the R variable to $R = 1$ and in the d variable to $d \in \{p_1, \ldots, p_M\}$. We then have

$$\sum_{j \leq M} \tilde{G}(p_j) \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \|Hg/p_j\| = O(1).$$

Note that since $p_j \neq 3$, we have $\tilde{G}(p_j) = G_3(1)\tilde{G}_{1,s}(p_j) \gg |F(p_j) - 1|^2 \geq \eta^2$. We select $H := \frac{1}{2}p_1 \cdots p_M$, again finding that $\|Hg/p_j\| = 1/2$ for all $1 \leq j \leq M$. Thus,

$$M\eta^2 \ll \sum_{1 \leq j \leq M} \tilde{G}(p_j) \ll \sum_{j \leq M} \tilde{G}(p_j) \sum_{g|\mathrm{rad}(q'/2^{v_2(q')})} \mu(g) g^{-2} \|Hg/p_j\| \ll 1.$$

Since, by assumption, $\mathcal{E}_\eta$ is infinite we can take M arbitrarily large, so this is a contradiction. Thus, $\mathcal{E}_\eta$ must be finite, as claimed. $\square$

The above argument fails to conclude directly that $|f(p)| < 1$ only finitely often. To accomplish this we need an additional result.

Proposition 3.5. *Let $f : \mathbb{N} \to \mathbb{U}$ be completely multiplicative, satisfying Property B'. Assume that $\{p : |f(p)| < 1\}$ is thin, and that there is a non-principal Dirichlet character χ such that f pretends to be χ. Then $|\{p : f(p) \neq \chi(p)\}| < \infty$.*

Proof. Put $\mathcal{S} := \{p : |f(p)| < 1\}$, and for $\eta > 0$ put $\mathcal{S}_\eta := \{p \in S : |1 - f(p)\overline{\chi}(p)| \leq \eta\}$. By part c) of Proposition 3.4, we know that $|\mathcal{S} \backslash \mathcal{S}_\eta| < \infty$ for each fixed $\eta > 0$. Taking $\varepsilon > 0$ sufficiently small, in what follows it is enough to show that $|\mathcal{S}_\eta| < \infty$ with $\eta = \varepsilon^2$.
Assume for the sake of contradiction that $|\mathcal{S}_{\varepsilon^2}| = \infty$. Let $H > q$. The goal of the proof will be to show that one can find two intervals $(x, x+H]$ and $(y, y+H]$ with $y > x > H$ such that

$$\left| \sum_{x < n \leq x+H} f(n) - \sum_{y \leq n \leq y+H} f(n) \right|$$

can be made arbitrarily large, as soon as H is large enough. In such a case, Property B' and the triangle inequality imply that

$$\left| \sum_{x < n \leq x+H} f(n) - \sum_{y < n \leq y+H} f(n) \right| \leq \left| \sum_{x < n \leq x+H} f(n) \right| + \left| \sum_{y < n \leq y+H} f(n) \right| = O(1), \tag{20}$$

a contradiction.

Let $1 \leq l \leq H$ be an integer parameter to be chosen. We define a polynomial

$$P_H(m) := \prod_{1 \leq r \leq H} ((H!)^2 m + r).$$

Since $\mathcal{S}_{\varepsilon^2}$ is infinite we can select l distinct primes $p_1, \ldots, p_l > H$, all of which belong to $\mathcal{S}_{\varepsilon^2}$. For fixed integers $1 \leq r_1, \ldots, r_l \leq H$, and $k_1, \ldots, k_l \in \mathbb{N}$ all to be chosen, consider the set

$$\mathcal{M}_{\boldsymbol{r}}(x) := \{m \leq x : p \in \mathcal{S}, p|P_H(m) \Longrightarrow p \leq H \text{ and } |f((H!)^2 m + r_j)| = 1 \ \forall 1 \leq j \leq l\},$$

and, for fixed primes $p_j > H$ and $k_j \in \mathbb{N}$ fixed for each $1 \leq j \leq l$ we define

$$\mathcal{N}_{\boldsymbol{r},\boldsymbol{k}}(x) := \{m \leq x : p_j^{k_j} || ((H!)^2 m + r_j), 1 \leq j \leq l \text{ and } p \in \mathcal{S}, p|P_H(m) \Longrightarrow p = p_j \text{ or } p \leq H\}.$$

The set $\mathcal{M}_{\boldsymbol{r}}(x)$ corresponds to choices of m such that $P_H(m)$ is free of large prime factors from $\mathcal{S}$ (and with certain prescribed integers yielding unimodular values of f), while $\mathcal{N}_{\boldsymbol{r},\boldsymbol{k}}(x)$ is an analogue of $\mathcal{M}_{\boldsymbol{r}}(x)$ in which the multiplicities of chosen large primes from $\mathcal{S}_{\varepsilon^2}$ are fixed.

Note that the condition $|f(n)| = 1$ is equivalent to $(n, \mathcal{S}) = 1$. Since

$$((H!)^2 m + r, \mathcal{S} \cap [1, H]) = (r, \mathcal{S} \cap [1, H]), \tag{21}$$

we pick $\boldsymbol{r}$ to consist of residue classes modulo H that are coprime to $\prod_{p \leq p, p \in \mathcal{S}} p$, in which case the same is true for $(H!)^2 m + r$ as well, for any m. Many such choices can be made for H sufficiently large, since $\sum_{p \in \mathcal{S}} 1/p < \infty$.

Let x be chosen large in terms of H and the primes $p_1, \ldots, p_l$. A lower bound sieve and (21) show that

$$\begin{aligned} |\mathcal{M}_{\boldsymbol{r}}(x)| &\geq |\{m \leq x : (P_H(m), \mathcal{S} \cap [H+1, \infty)) = 1, ((H!)^2 m + r_j, \mathcal{S} \cap [1, H]) = 1 \text{ for all } j\}| \\ &\gg x \prod_{\substack{p > H \\ p \in \mathcal{S}}} \left(1 - \frac{H}{p}\right) \gg x \exp\left(-H \sum_{\substack{p > H \\ p \in \mathcal{S}}} 1/p\right) \gg_H x. \end{aligned}$$

Similarly, the lower bound sieve together with the Chinese remainder theorem shows that

$$\begin{aligned} &|\mathcal{N}_{\boldsymbol{r},\boldsymbol{k}}(x)| \\ &\geq |\{m \leq x : (H!)^2 m + r_j \equiv p_j^{k_j} \pmod{p_j^{k_j+1}} \text{ and } (\prod_{\substack{1 \leq r \leq H \\ r \neq r_1, \ldots, r_l}} ((H!)^2 m + r), \mathcal{S} \cap [H+1, \infty)) = 1\}| \\ &\gg_l \frac{x}{p_1^{k_1} \cdots p_l^{k_l}} \exp\left(-H \sum_{\substack{p > H \\ p \in \mathcal{S}}} 1/p\right) \gg_{H,l} x. \end{aligned}$$

Thus, provided x is large enough we can find elements of $\mathcal{M}_{\boldsymbol{r}}(x)$ and $\mathcal{N}_{\boldsymbol{r},\boldsymbol{k}}(x)$ as large as desired.

Fix a tuple $\boldsymbol{r}$ as described above and let $m_0 \in \mathcal{M}_{\boldsymbol{r}}(x)$. Let l be large in terms of $\varepsilon > 0$. By the pigeonhole principle, there is a point $z \in S^1$ and a set $\mathcal{G} \subseteq \{1, \ldots, l\}$ of size $\gg \varepsilon l$ such that for all $j \in \mathcal{G}$ we have

$$|f((H!)^2 m_0 + r_j) - z| \leq 10\varepsilon.$$

We select k_j large enough that $|f(p_j^{k_j})| < 1/2$ when $j \in \mathcal{G}$, while if $j \notin \mathcal{G}$ then we put $k_j = 1$. With these choices made, we then select $\tilde{m}_0 \in \mathcal{N}_{\boldsymbol{r},\boldsymbol{k}}(x)$, with $\tilde{m}_0 > m_0$.

We now observe the following. Let $1 \leq r \leq H$ with $r \neq r_j$ for any $1 \leq j \leq l$.

For such r and for $m \in \mathbb{N}$ put

$$S_r(m) := \prod_{p \in \mathcal{S}} p^{v_p((H!)^2 m + r)}.$$

Note that when $m \in \mathcal{M}_{\boldsymbol{r}}(x)$, the number $S_r(m)$ depends only on primes dividing r. Moreover, as $r \leq H$ and $v_p(S_r(m)) \leq v_p(r)$ for all $p \leq H$, we deduce that $S_r(m)$ divides $H!$. We use this fact below. Factoring $(H!)^2 \tilde{m}_0 + r$, we get

$$f((H!)^2 \tilde{m}_0 + r) = f(S_r(\tilde{m}_0)) \cdot f\left(\tilde{m}_0 \frac{(H!)^2}{S_r(\tilde{m}_0)} + \frac{r}{S_r(\tilde{m}_0)}\right).$$

If $p \in \mathcal{S}$ then $v_p((H!)^2 m_0 + r) = v_p(r) = v_p((H!)^2 \tilde{m}_0 + r)$, and thus $S_r(m_0) = S_r(\tilde{m}_0)$. As $S_r(m)|H!$ for $m \in \{m_0, \tilde{m}_0\}$, we have $(H!)^2/S_r(m) \equiv 0 \pmod q$ for H large enough, and thus

$$\frac{(H!)^2}{S_r(\tilde{m}_0)}\tilde{m}_0 + \frac{r}{S_r(\tilde{m}_0)} \equiv \frac{(H!)^2}{S_r(m_0)}m_0 + \frac{r}{S_r(m_0)} \pmod q.$$

By applying Proposition 3.4a) and the fact that $(((H!)^2 m_0 + r)/S_r(m_0), \mathcal{S}) = 1$, it follows that for all $r \neq r_j$ for all $1 \leq j \leq l$,

$$\begin{aligned} f((H!)^2\tilde{m}_0 + r) &= f(S_r(\tilde{m}_0))\chi\left(\frac{(H!)^2}{S_r(\tilde{m}_0)}\tilde{m}_0 + \frac{r}{S_r(\tilde{m}_0)}\right) \\ &= f(S_r(m_0))\chi\left(\frac{(H!)^2}{S_r(m_0)}m_0 + \frac{r}{S_r(m_0)}\right) = f((H!)^2 m_0 + r). \end{aligned}$$

Next, suppose $r = r_j$ for some $1 \leq j \leq l$. By a similar argument, we can show that

$$f((H!)^2\tilde{m}_0 + r_j) = (f(p_j)\overline{\chi}(p_j))^{k_j}\, f((H!)^2 m_0 + r_j),$$

for all $1 \leq j \leq l$, since then $S_{r_j}(\tilde{m}_0) = p_j^{k_j} S_{r_j}(m_0)$.

We are now ready to show the desired contradiction. For $m \geq 1$ put $I_{m,H} := [(H!)^2 m + 1, (H!)^2 m + H]$. Then (20) with $y = (H!)^2\tilde{m}_0$ and $x = (H!)^2 m_0$ shows that

$$\left|\sum_{n \in I_{m_0,H}} f(n) - \sum_{n \in I_{\tilde{m}_0,H}} f(n)\right| = O(1). \tag{22}$$

In light of the above calculation, we thus deduce

$$\left|\sum_{1 \leq j \leq l}(1 - (f\overline{\chi}(p_j))^{k_j}) f((H!)^2 m_0 + r_j)\right| = \left|\sum_{n \in I_{\tilde{m}_0,H}} f(n) - \sum_{n \in I_{m_0,H}} f(n)\right| = O(1),$$

uniformly in H and l.

We now split the set of $1 \leq j \leq l$ into the sets $\mathcal{G}$ and $\mathcal{B} := \{1, \ldots, l\} \backslash \mathcal{G}$. Recall that if $j \in \mathcal{G}$ then $|f(p_j)^{k_j}| < 1/2$, while if $j \in \mathcal{B}$ then $k_j = 1$. Thus, by the triangle inequality,

$$\left|\sum_{1 \leq j \leq l}(1 - (f\overline{\chi})(p_j)^{k_j}) f((H!)^2 m_0 + r_j)\right| \geq \left|\sum_{j \in \mathcal{G}}(1 - (f\overline{\chi})(p_j)^{k_j}) f((H!)^2 m_0 + r_j)\right| - \sum_{j \in \mathcal{B}} |1 - (f\overline{\chi})(p_j)|.$$

Now we recall that there is a $z \in S^1$ such that $\max_{j \in \mathcal{G}} |f((H!)^2 m_0 + r_j) - z| \leq 10\varepsilon$, and so as $|\mathcal{G}| \gg \varepsilon l$ the first term above is

$$\left|\sum_{j \in \mathcal{G}}(z + 10\theta_j\epsilon)(1 - (f\overline{\chi})(p_j)^{k_j})\right| > |\mathcal{G}|\,(1/2 - 20\epsilon) \gg \varepsilon l$$

for some $|\theta_j| \leq 1$. On the other hand, since $p_j \in \mathcal{S}_{\varepsilon^2}$ for all j we have $|1 - (f\overline{\chi})(p_j)| \leq \varepsilon^2$ for all j and thus

$$\sum_{\substack{1 \leq j \leq l \\ j \in \mathcal{B}}} |1 - f\overline{\chi}(p_j)| \leq \varepsilon^2 l.$$

Picking ε sufficiently small, we see that

$$\left|\sum_{n \in I_{m_0,H}} f(n) - \sum_{n \in I_{\tilde{m}_0,H}} f(n)\right| \gg \varepsilon l - \varepsilon^2 l \gg \varepsilon l.$$

Comparing this bound with (22), we conclude that $l = O(1/\varepsilon)$, which is a contradiction since we can take l arbitrarily large in terms of ε since $\mathcal{S}_{\varepsilon^2}$ was assumed infinite.

It follows that $|\mathcal{S}_{\varepsilon^2}| < \infty$ as well, as required. □

Proof of Theorem 1.3. ($\Rightarrow$) Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function satisfying Property B and such that $\{p : |f(p)| < 1\}$ is thin.

By Lemma 3.1, f takes values in $\mathbb{U}$, and by Lemma 3.2 there is a $t \in \mathbb{R}$ and a primitive Dirichlet character χ modulo $q > 1$ such that f pretends to be $\chi(n)n^{it}$.

Set $\tilde{f}(n) := f(n)n^{-it}$ for all $n \in \mathbb{N}$. By Lemma 3.3, $\tilde{f}$ satisfies Property B'. Applying Proposition 3.4 a) and b) to $\tilde{f}$, if $p \nmid q$ and $|\tilde{f}(p)| = |f(p)| = 1$ then $\tilde{f}(p) = \chi(p)$, i.e., $f(p) = \chi(p)p^{it}$, while if $p|q$ then $|f(p)| < 1$.
Furthermore, according to Proposition 3.5, the set of primes p for which $|f(p)| < 1$ is finite. Thus, there is a finite set S that includes the set of primes $p|q$ such that if $p \in S$ then $|f(p)| < 1$ while if $p \notin S$ then $f(p) = \chi(p)p^{it}$.

($\Leftarrow$) Assume next that there is a $t \in \mathbb{R}$, a primitive Dirichlet character χ of conductor $q > 1$ and a finite set of primes S containing the primes $p|q$ such that if $p \in S$ then $|f(p)| < 1$, and otherwise $f(p) = \chi(p)p^{it}$. We claim that such a function has bounded partial sums.

To see this, let us note first of all that functions of the form $n \mapsto \chi(n)n^{it}$ have bounded partial sums for any fixed t. When $t = 0$ this follows from the orthogonality of Dirichlet characters, so assume that $t \neq 0$. In this case,

$$\begin{aligned}\sum_{n \le x} \chi(n)n^{it} &= \sideset{}{^*}\sum_{a \pmod q} \chi(a) \sum_{m \le x/q} (mq + a)^{it} + O_q(1) \\ &= q^{it} \sideset{}{^*}\sum_{a \pmod q} \chi(a) \sum_{m \le x/q} m^{it}(1 + a/(qm))^{it} + O_q(1).\end{aligned}$$

By a Taylor approximation, for m sufficiently large in terms of $|t|$ we have

$$(1 + a/(qm))^{it} = 1 + \frac{ita}{qm} + O_t\left(\frac{a^2}{q^2m^2}\right),$$

so that the sum above is

$$q^{it} \sideset{}{^*}\sum_{a \pmod q} \chi(a) \sum_{m \le x/q} m^{it}\left(1 + \frac{ita}{qm}\right) + O_{q,t}(1) = itq^{-1+it} \sideset{}{^*}\sum_{a \pmod q} a\chi(a) \sum_{m \le x/q} \frac{1}{m^{1-it}} + O_{q,t}(1).$$

Since $q = O(1)$, we easily find that the inner sum is precisely $\zeta(1 + 1/\log x - it) + O_q(1) = O_{q,t}(1)$ whenever $t \neq 0$. Hence, we have

$$\sum_{n \le x} \chi(n)n^{it} = O_{q,t}(1). \tag{23}$$

Since $q > 1$ we know that $S \neq \emptyset$. Write $S = \{p_1, \ldots, p_N\}$, where $N := |S|$ so that the p_j are distinct; note that by assumption, if m is coprime to the primes of S then $f(m) = \chi(m)m^{it} \neq 0$. For any x large enough we have

$$\begin{aligned}\sum_{n \le x} f(n) &= \sum_{k_1,\ldots,k_N \ge 0} f(p_1)^{k_1} \cdots f(p_N)^{k_N} \sum_{\substack{m \le x/\prod_i p_i^{k_i} \\ (m,S)=1}} \chi(m)m^{it} \\ &= \sum_{k_1,\ldots,k_N \ge 0} \prod_{1 \le j \le N} f(p_j)^{k_j} \sum_{d|p_1\cdots p_N} \mu(d)\chi(d)d^{it} \sum_{m \le x/(dp_1^{k_1}\cdots p_N^{k_N})} \chi(m)m^{it}.\end{aligned}$$

Bounding the inner sum using (23), we obtain

$$\left|\sum_{n\le x} f(n)\right| \ll_{q,t} 2^N \prod_{1\le j\le N}\sum_{k_j\ge 0}|f(p_j)|^{k_j} \ll_S \prod_{1\le j\le N}(1-|f(p_j)|)^{-1},$$

which is a bounded product. Thus, any f of this shape has bounded partial sums, as claimed. $\square$

Proof of Corollary 1.4 b). Suppose $f:\mathbb{N}\to\mathbb{U}$ is a completely multiplicative function with bounded partial sums, such that $|f(p)|=1$ for all but a finite (possibly empty) set S of primes, such that $p\in S$ implies that $f(p)=0$. Clearly, then, the set $\{p:|f(p)|<1\}$ is finite, thus thin, so by Theorem 1.3 there is a primitive character χ modulo q and a $t\in\mathbb{R}$ such that for all but a finite set S' of primes p we have $f(p)=\chi(p)p^{it}$. For those primes $p\in S'$ we have $|f(p)|<1$, which implies that $f(p)=0$. Thus, $S'\subseteq S$. Letting $q':=[q,\prod_{p\in S'}p]$ and replacing χ by $\psi:=\chi\chi_0^{(q')}$, where $\chi_0^{(q')}$ denotes the principal character modulo q', we see that $f(n)=\psi(n)n^{it}$ for all n, as required. $\square$

4. Examples where $f(p)$ vanishes at many primes

Here we present some examples of completely multiplicative functions f for which $\{p:|f(p)|<1\}$ is not thin, and that look nothing like twisted Dirichlet characters.

i) Let p be a fixed prime and let $k\ge 2$. Let ζ be a fixed primitive kth root of unity, and define a completely multiplicative function by $f(p)=\zeta$, $f(p')=0$ for all $p'\neq p$. We then clearly have

$$\sum_{n\le x} f(n)=\sum_{l\le \log x/\log p}\zeta^l=\sum_{b\le k\{\frac{1}{k}\lfloor \log x/\log p\rfloor\}}\zeta^b=O_k(1),$$

using the orthogonality relation $\sum_{b \pmod k}\zeta^k=0$ to establish the second to last equality. More generally, one may take any $r>1$ and $\alpha\in(0,1)$, choose $f(p)=e^{2\pi i\alpha}$ and select $|f(q)|\le\frac{1}{q^r}$ for all other primes $p\neq q$. Following the same lines as above we conclude that f has bounded partial sums.

ii) Further we observe that partial sums can be extremely small. Take $f(n):=\lambda(n)/n$ for all n, which is clearly completely multiplicative with $|f(p)|=1/p<1$ for all p. Applying the prime number theorem and partial summation, it is easy to check that for x sufficiently large,

$$\left|\sum_{n\le x} f(n)\right| \ll_A (\log x)^{-A},$$

which establishes the rapid decay of the partial sums towards zero in this case.

5. Proof of Theorem 1.5 and Further Applications

In this section, we consider two further applications of the "rotation trick" that illustrate how this technique is applicable to multiplicative functions that are not necessarily completely multiplicative. To begin with, we prove the squarefree discrepancy conjecture (Theorem 1.5). This is done in two steps, the first of which is the following proposition.

Proposition 5.1. *Let $g:\mathbb{N}\to\{-1,+1\}$ be a multiplicative function, and let $f=\mu^2 g$. Assume that f pretends to be a real quadratic character χ modulo q, and let $S:=\{p:f(p)\neq$*

$\chi(p)\}$. *If*

$$\limsup_{x\to\infty}\left|\sum_{n\le x} f(n)\right| < \infty$$

then $|S| < \infty$.

Proof. Note that $f(n) = \mu^2(n)\prod_{p|n} g(p)$, so we may assume without loss of generality that g is completely multiplicative.
Let H be sufficiently large in terms of q, and for $m \in \mathbb{N}$ set $I_H(m) := [(H!)^2m+1, (H!)^2m+H]$ as above. Let $1 \le \ell \le H$, fix an ℓ-tuple $\boldsymbol{r} = (r_1,\dots,r_\ell) \in [1,H]$ such that
i) $\mu^2(r_j) = 1$ for all $1 \le j \le \ell$
ii) $p|r_j \Rightarrow p \notin S$, for all $1 \le j \le \ell$;
such a choice of tuples, for H large enough in terms of l, follows from a lower bound sieve. By the pigeonhole principle we can then select a subset $\{r_1',\dots,r_t'\}$ of these such that $g(r_j') = g(r_k')$ for all $1 \le j,k \le t$, with $t \ge \ell/2$.
Assume that $|S| = \infty$. Then we can pick distinct primes $p_1,\dots,p_t > H$ with $p_j \in S$ for all $1 \le j \le t$. With the t-tuple $\boldsymbol{r}'$ chosen above and x chosen large as a function of H, we define the set

$$\mathcal{M}_{\boldsymbol{r}'}(x) := \left\{m \le x : \left(p \in S, p|\prod_{n\in I_H(m)} n \Rightarrow p \le H\right) \text{ and } \mu^2((H!)^2m+r) = \begin{cases}1 & \text{if } r = r_j' \\ 0 & \text{if } r \ne r_j'.\end{cases}\right\}$$

By a lower bound sieve argument (similar to the proof of Proposition 3.5), we can show that $\mathcal{M}_{\boldsymbol{r}}(x) \gg_H x$.

We may also define $\mathcal{N}_{\boldsymbol{r}}(x)$ to be the set of $m \le x$ satisfying the following properties:
a) if $p \in S$ and $p|\prod_{\substack{1\le r\le H\\ r\ne r_j'}}((H!)^2m+r)$ then $p \le H$
b) for each $1 \le j \le t$, $(H!)^2m + r_j' \equiv p_j \pmod{p_j^2}$
c) if $r \ne r_j'$ for all $1 \le j \le t$ then $\mu^2((H!)^2m+r) = 0$, whereas if $r = r_j'$ for some $1 \le j \le t$ then $\mu^2((H!)^2m+r) = 1$.
Once again, this set satisfies $\mathcal{N}_{\boldsymbol{r}}(x) \gg_{p_j,H} x$ by the chinese remainder theorem and a lower bound sieve.
We now pick $m' \in \mathcal{N}_{\boldsymbol{r}}(x)$ and $m \in \mathcal{M}_{\boldsymbol{r}}(x)$ with $m' > m$. Arguing as in the proof of Proposition 3.5, we note that $((H!)^2m + r_j', \mathcal{S}) = (r_j', \mathcal{S}\cap[1,H])$, so that if we set

$$S_r(m) := \prod_{\substack{p^k||(H!)^2m+r\\ p\in S}} p^k$$

then $S_{r_j'}(m)|r_j'$ and $S_{r_j'}(m)|H!$, and so

$$g((H!)^2m+r_j') = g(S_{r_j'}(m))\chi\left(\frac{(H!)^2}{S_{r_j'}(m)}m + \frac{r_j'}{S_{r_j'}(m)}\right) = g(S_{r_j'}(m))\chi(r_j'/S_{r_j'}(m)) = g(r_j'),$$

for all $1 \le j \le t$. On the other hand, as $((H!)^2m' + r_j', \mathcal{S}) = p_j(r_j', \mathcal{S}\cap[1,H])$, we have $S_{r_j'}(m') = p_jS_{r_j'}(m)$, and so

$$\begin{aligned} g((H!)^2m'+r_j') &= g(p_j)g(S_{r_j'}(m))\chi\left(\frac{(H!)^2m'+r_j'}{p_jS_{r_j'}(m)}\right) = g\chi(p_j)g(S_{r_j'}(m))\chi\left(\frac{(H!)^2}{S_{r_j'}(m)}m + \frac{r_j'}{S_{r_j'}(m)}\right) \\ &= g\chi(p_j)g(S_{r_j'}(m))\chi(r_j'/S_{r_j'}(m)) = g\chi(p_j)g(r_j'). \end{aligned}$$

It follows, then, that

$$\begin{aligned}\left|\sum_{n\in I_H(m')} f(n) - \sum_{n\in I_H(m)} f(n)\right| &= \left|\sum_{\substack{n\in I_H(m')\\ \mu^2(n)=1}} g(n) - \sum_{\substack{n\in I_H(m)\\ \mu^2(n)=1}} g(n)\right| \\ &= \left|\sum_{1\le j\le t} (g((H!)^2m' + r_j') - g((H!)^2m + r_j'))\right| \\ &= \left|\sum_{1\le j\le t} (1 - (g\chi(p_j)))g(r_j')\right| = 2t \ge \ell\end{aligned}$$

given that $g(r_j)$ has the same sign for all $1 \le j \le t$, and $g\chi(p_j) = -1$ on S. Now since H can be taken large, ℓ can also be chosen as large as desired; on the other hand, we have

$$\left|\sum_{n\in I_H(m')} f(n) - \sum_{n\in I_H(m)} f(n)\right| \le 4 \max_{1\le N\le 2(H!)^2x} \left|\sum_{n\le N} f(n)\right| \ll 1,$$

as x gets large. This is a contradiction, and the claim that $|S| < \infty$ follows. □

Proof of Theorem 1.5. Assume for the sake of contradiction that there is a multiplicative function $g : \mathbb{N} \to \{-1, +1\}$ such that for $f := \mu^2 g$ we have

$$\limsup_{x\to\infty} \left|\sum_{n\le x} f(n)\right| < \infty. \tag{24}$$

By [1, Theorem 1.1], there is a real, primitive Dirichlet character χ modulo q, where $(q, 6) = 1$, such that f pretends to be χ, i.e., $\sum_p (1 - f(p)\chi(p))/p < \infty$, and such that $f(2)\chi(2) = f(3)\chi(3) = -1$. Furthermore, by Proposition 5.1 the set $S := \{p : f(p)\chi(p) = -1\}$ is finite. Put $q' := [q, \prod_{p\in S} p]$, so that all the primes p satisfying $f(p)\chi(p) \neq 1$ satisfy $p \mid q'$.

Consider the Dirichlet series $F(s) := \sum_{n\ge 1} f(n)/n^s$ of f, which defines an analytic function in the region $\mathrm{Re}(s) > 1$. Also, put $M_f(x) := \sum_{n\le x} f(n)$. We will show that $|M_f(x)| = O(x^{1/4-\varepsilon})$ cannot hold for any fixed $\varepsilon > 0$.

By partial summation, we have

$$F(s) = \int_1^\infty M_f(x)x^{-s-1}dx,$$

initially for $\mathrm{Re}(s) > 1$, but if $M_f(x) = O(x^{1/4-\varepsilon})$, then this formula extends F analytically to the half-plane $\mathrm{Re}(s) > 1/4 - \varepsilon$.

Now, comparing Euler products when $\mathrm{Re}(s) > 1$, we note that since $\chi(p)^2 = 1$ for all $p \nmid q'$,

$$\begin{aligned}F(s) &= \prod_p \left(1 + g(p)p^{-s}\right) = \prod_{p\mid q'}(1 + g(p)p^{-s}) \prod_{p\nmid q'}(1 + \chi(p)p^{-s}) \\ &= \prod_{p\mid q'}(1 + g(p)p^{-s}) \prod_{p\nmid q'}(1 - p^{-2s})(1 - \chi(p)p^{-s})^{-1} \\ &= \prod_{p\mid q'}(1 + g(p)p^{-s})(1 - \chi(p)p^{-s})(1 - p^{-2s})^{-1} L(s,\chi)\zeta(2s)^{-1} \\ &=: P(s)L(s,\chi)\zeta(2s)^{-1}.\end{aligned}$$

Note that $P(s)$ has no zeros in $\mathrm{Re}(s) > 0$, so by analytic continuation we see that $L(s,\chi)\zeta(2s)^{-1} = F(s)/P(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 1/4 - \varepsilon$.

By an old result of Hardy and Littlewood [6], we can find $\gg T$ real numbers $\gamma \in [-T, T]$ such that $\zeta(1/2 + 2i\gamma) = 0$. Thus, there are $\gg T$ points $\rho = 1/4 + i\gamma$ at which $\zeta(2\rho) = 0$. Since $L(s,\chi)/\zeta(2s)$ must be analytic along $\mathrm{Re}(s) = 1/4$, we have $L(\rho,\chi) = 0$ for these $\gg T$ points ρ, and thus

$$|\{(\sigma,t) \in (0,1/4] \times [-T,T] : L(\sigma+it,\chi) = 0\}| \gg T. \tag{25}$$

On the other hand, since χ is real, the functional equation gives that $L(\sigma+it,\chi) = 0$ if and only if $L(1-\sigma-it,\chi) = 0$ whenever $\sigma \in (0,1)$, and by Huxley's zero density estimate [7] we have

$$\begin{aligned}|\{(\sigma,t) \in (0,1/4] \times [-T,T] : L(\sigma+it,\chi) = 0\}| &= |\{(\sigma,t) \in [3/4,1) \times [-T,T] : L(\sigma+it,\chi) = 0\}|\\ &\ll_\eta (qT)^{(12/5+\eta)(1-3/4)} \ll T^{0.7} = o(T),\end{aligned}$$

which contradicts (25). (Note that we only needed a very weak saving here, and thus also some zero density estimates older than Huxley's would have sufficed.)

It follows that $F(s)$ is not analytic in $\mathrm{Re}(s) > 1/4 - \varepsilon$ if $\varepsilon > 0$, and therefore $|M_f(x)| \gg x^{1/4-\varepsilon}$ for infinitely many integers $x \geq 1$, which certainly contradicts our initial assumption (24) about boundedness of partial sums of f. □

As a further example application concerning not necessarily completely multiplicative functions, we consider the problem of characterizing multiplicative functions $f : \mathbb{N} \to \{-1,+1\}$ (that are not necessarily completely multiplicative) satisfying

$$\sum_{n \leq x} f(n) = O(1). \tag{26}$$

This was the content of the so-called Erdős–Coons–Tao conjecture, and was fully resolved in [8] based on the preliminary work of Tao [13]. Here we outline how the methods of the current paper can be used to give a very short proof of a weaker statement.

Theorem 5.2. *Let $f : \mathbb{N} \to \{-1,1\}$ be a multiplicative function such that* (26) *holds. Then, there exists a primitive Dirichlet character χ, such that $f(p^k) = \chi(p^k)$ for all but finitely many values of p and $k \geq 1$.*

Outline of the proof of Theorem 5.2. The same argument as in the proof of Lemma 3.2 yields the existence of $t \in \mathbb{R}$ and a primitive Dirichlet character χ of conductor $q > 1$ such that

$$\sum_p \frac{1 - \mathrm{Re}(f(p)\overline{\chi}(p)p^{-it})}{p} < \infty.$$

Since f is real-valued we may assume that $t = 0$ and that χ is a real character (this follows e.g., from Corollary 3.2 of [2]). We now proceed in exactly the same way as before. Fix large $H \geq 1$ and assume that there exists prime powers $p_1^{k_1} < p_2^{k_2} <, \dots, < p_H^{k_H}$ with $p_i > H$ and $f(p_i^{k_i}) = -\chi(p_i^{k_i})$. We then repeat the same arguments as in the proofs of Proposition 3.5 and Theorem 1.5 with primes p_i replaced with prime powers $p_i^{k_i}$ to obtain a contradiction if H is sufficiently large. □

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Max Planck Institute for Mathematics, Bonn, Germany
E-mail address: lklurman@gmail.com

Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada
E-mail address: smangerel@gmail.com

Department of Mathematics, California Institute of Technology, Pasadena, California, USA
E-mail address: apohoata@caltech.edu

Mathematical Institute, University of Oxford, Oxford, UK
E-mail address: joni.teravainen@maths.ox.ac.uk